Factorization of Natural Numbers
Considerations arising from a representation

1. Definition

The fundamental theorem of arithmetic states that every natural number $m>1$ is uniquely factored in the form:

$$m=p_1^{a_1}\cdot p_2^{a_2}\ldots\cdot p_k^{a_k}$$

where $p_1, p_2, \ldots p_k > 1$ are the prime factors (normally distinct from each other) and $a_1, a_2, \ldots a_k > 0$ the respective natural exponents.

In order to know the *degree of decomposition* of *m*, or on how many parts is composed *m* with respect to the operation *product*, it is defined **order of factoring of *m*, *Ω (m)*** (refer to Almost Prime), the sum of all the exponents, here understood as the number of occurrences of each factor:

$$\Omega(m) \equiv a_1+a_2+\ldots a_k$$

in Tab. 1 the first values of $\Omega(m)$ $1< m < 100$.

m	Ω(m)	m	Ω(m)	m	Ω(m)	m	Ω(m)	m	Ω(m)
2	1	22	2	42	3	62	2	82	2
3	1	23	1	43	1	63	3	83	1
4	2	24	4	44	3	64	6	84	4
5	1	25	2	45	3	65	2	85	2
6	2	26	2	46	2	66	3	86	2
7	1	27	3	47	1	67	1	87	2
8	3	28	3	48	5	68	3	88	4
9	2	29	1	49	2	69	2	89	1
10	2	30	3	50	3	70	3	90	4
11	1	31	1	51	2	71	1	91	2
12	3	32	5	52	3	72	5	92	3
13	1	33	2	53	1	73	1	93	2
14	2	34	2	54	4	74	2	94	2
15	2	35	2	55	2	75	3	95	2
16	4	36	4	56	4	76	3	96	6
17	1	37	1	57	2	77	2	97	1
18	3	38	2	58	2	78	3	98	3
19	1	39	2	59	1	79	1	99	3
20	3	40	4	60	4	80	5	…..	…..
21	2	41	1	61	1	81	4	…..	…..

Tab. 1

			99		
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				6	
					5
				4	
					3
					2
6° o.f.	***5° o.f.***	***4° o.f.***	***3° o.f.***	***2° o.f.***	***1° o.f.***

Tab. 2

2. Representation

In Tab. 2 the numbers in Tab. 1 are distributed in a row-column structure in which the natural numbers are associated with the rows and the columns identify for each *m*, from the right to the left, the factoring orders $\Omega(m)\ 1 < m < 100$.

In this way the natural numbers (> *1*) are identified, not only with their natural sequence, but also with a second parameter and can be organized and grouped into classes (i.e. the same order of factoring).

Each number *m* can be factored, at most, as a power of *2*. Consequently the highest class *m* can occupy is $\log_2 m$; more precisely $floor(\log_2 m)$, where *floor* is the whole lower part (sometimes indicated enclosed in ⌊⌋), that is the nearest integer ≤.

The first column below *(1st order of factorization)* is the class prime numbers, for example the fourth column *(4th order of factorization)* is the class of the fourth numbers and starts with *16, 24 and 36* because $16 = 2^4$ i.e. $\Omega(16) = 4$, $24 = 2^3 \cdot 3^1$ i.e. $\Omega(24) = 4$ and $36 = 2^2 \cdot 3^2$ i.e. $\Omega(36) = 4$. So for the numbers of the other classes.

Therefore: the second numbers, the third numbers, fourth, fifth, sixth, ... n-th ...

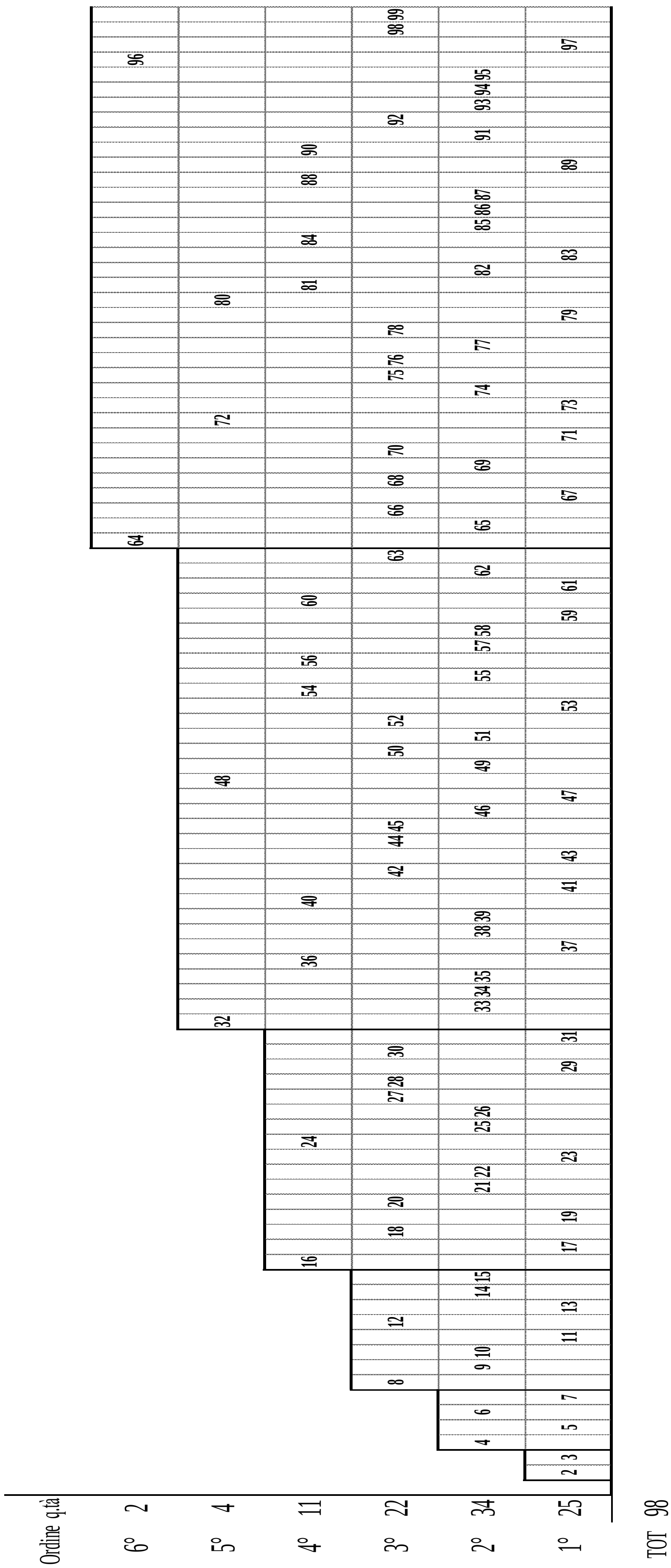


Fig. 1

(Note. For reasons of limited space, Fig. 1 is positioned vertically, therefore the rows go from the bottom to the top)

In order to better visualize the structure of Tab. 2, in Fig.1 it is represented, with a ***boxes*** diagram (unit positions), the distribution of the numbers from *2* to *99*.

Each level (row) identifies a factorization order *k* (from here *o.f.*) and starts with the value 2^k for a some *k*.

Fig. 1 shows a ***step*** pattern for the distribution: rectangles with base 2^i and height *i* (as the *o.f.* involved):

- the *i-th* step has the first value 2^i, it contains 2^i numbers and it develops, obviously, over $2^i \cdot i$ ***boxes***; the density of numbers compared to the ***boxes*** is:

$$d=2^i/(2^i \cdot i)=1/i$$

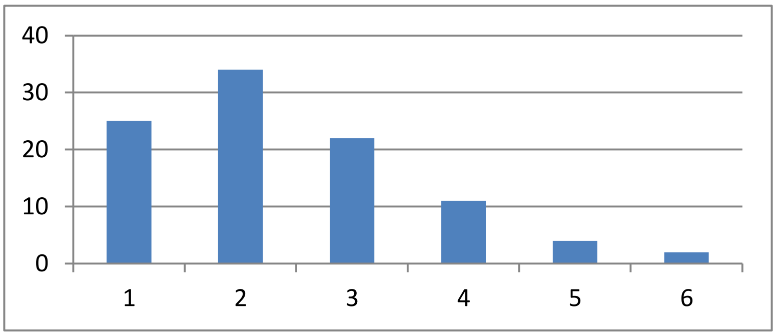
- the *i-th* step is structured on *i* ***levels*** (as many as *o.f.*)
- in each step the upper level contains only two numbers 2^i e $2^{i-1} \cdot 3$
- the basic level of each step (relative to the *1st o.f.*) contains only prime numbers.

This representation is a two-dimensional enlargement of the classical one which represents the natural numbers as a sequence in unitary positions:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99

Assumption: at least two numbers are present on each level of each step.

To the left of the ordinate axis are indicated the *o.f.* and the relative quantities of natural numbers $1 < m < 100$: the prime numbers are *25*, the seconds numbers *34*, the third numbers *22*, the quarter numbers *11*, the fifths numbers *4*, the sixth numbers *2*



The quantities tend to decrease because, with $n = 99$, the highest levels are less populated.

The graph in Fig.2 represents the cardinality growth of each class *[k]* of numbers (*<100*) with identical *o.f.* (the figure shows the growth of the classes related to the first *6 o.f.*).

The growth of cardinality *[1]*, quantity of the numbers of the *1st* order (prime numbers), is represented by the double line; it is exceeded, between *25* and *33*, by the growth of those of cardinality *[2]*.

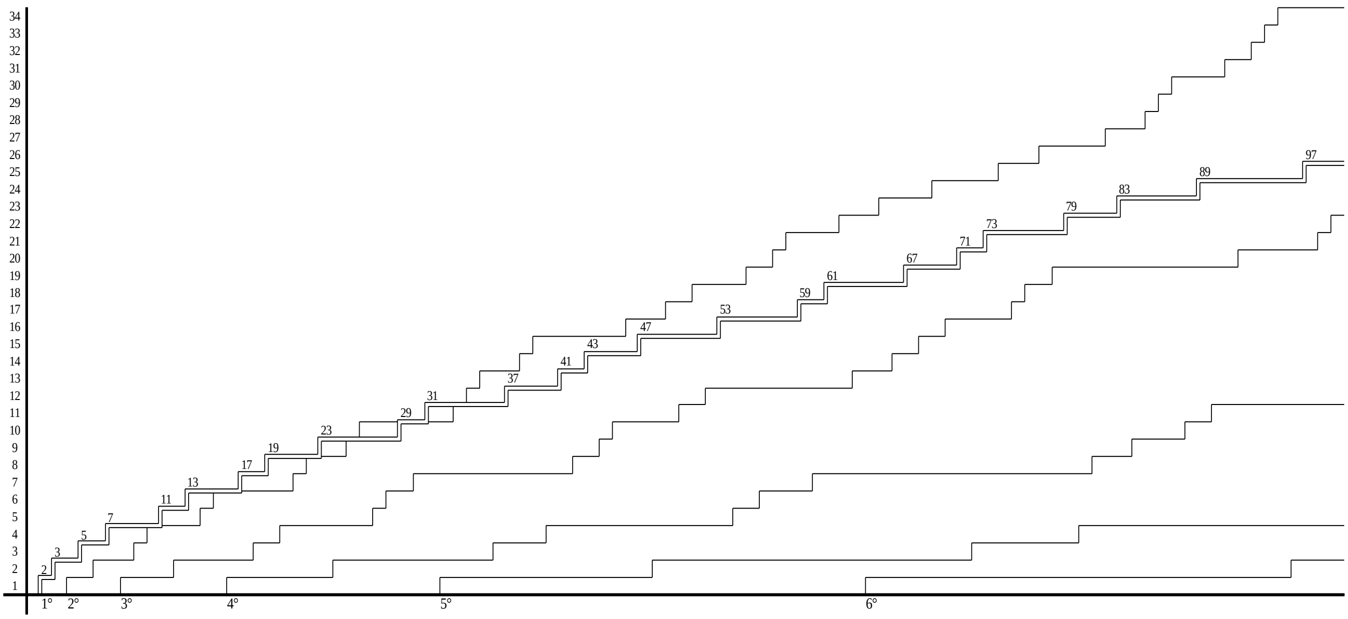


Fig. 2

Fig. 3 shows <u>some sections</u> of the distribution of the numbers from *2* to *n = 3500* in the various classes *[k], 1 ≤ k ≤ floor (log2 n)*.

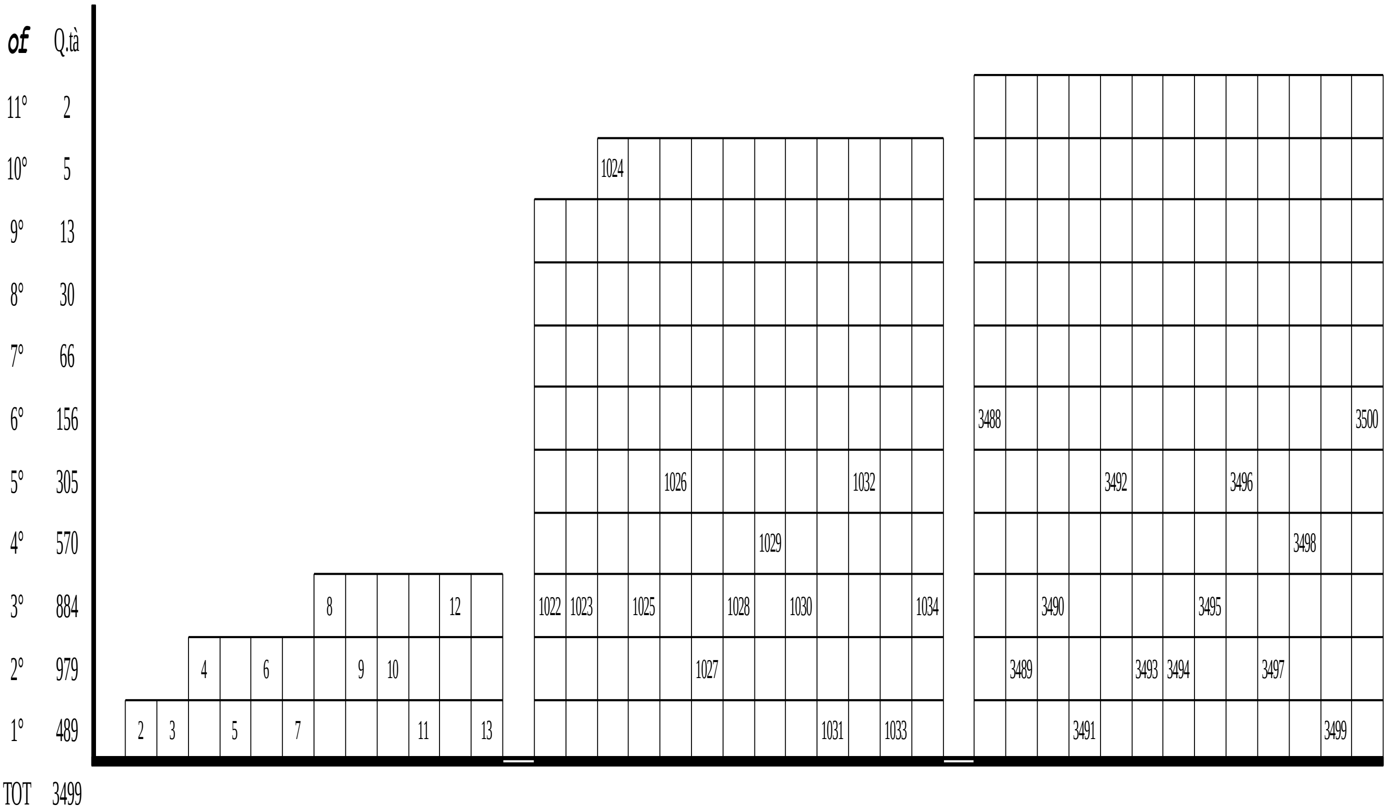


Fig. 3

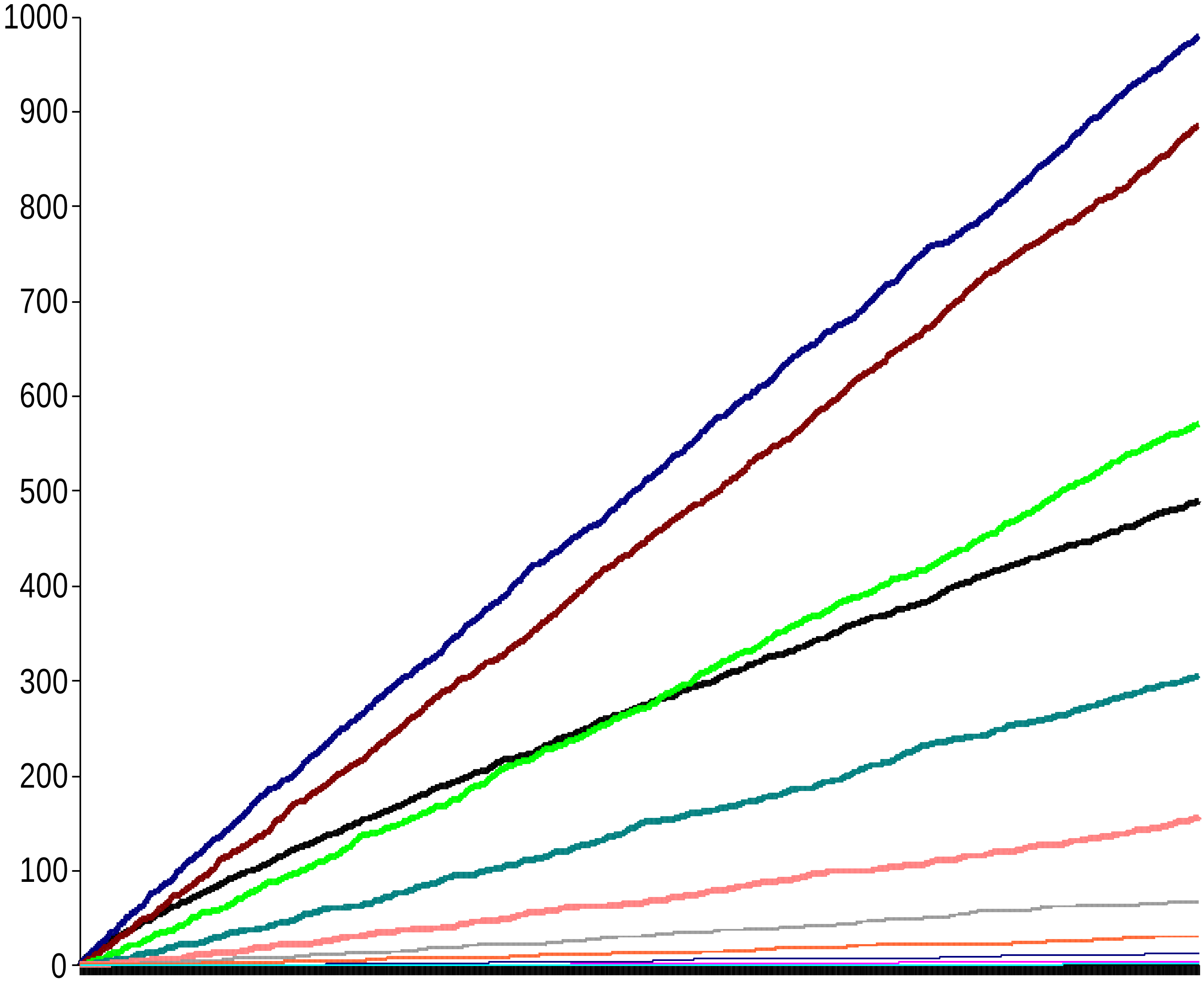


Fig. 4

The graph below shows the quantities relating to each of the *11* classes.

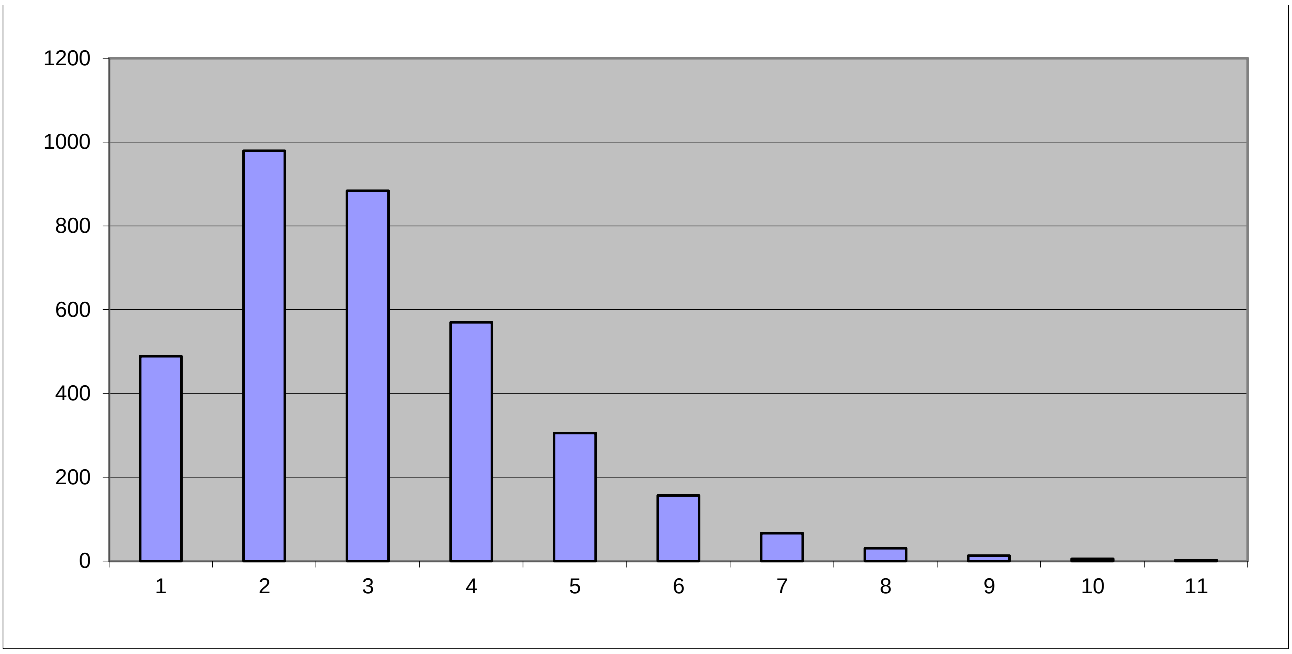


Fig. 4 graphically displays the cardinality growth of the first *11* classes $[k],\ 1 \le k \le 11$, each in relation to the corresponding *o.f.*, for the numbers from *2* to $n = 3500$ (as in Fig. 3):

- the trend of these cardinalities is a range of divergent lines (so it seems) representing the classes relating to *o.f.* from the *2nd* (the first up) to the line of the eleventh order (the last lines do not stand out well);
- the line representing the growth of prime numbers is black; it is immediately overcome by the *2nd* order line and, afterwards, by the lines of the *3rd* and *4th* order;
- the *2nd* order line exceeds that of the prime numbers between *25* and *33*;
- the *3rd* order line exceeds that of the prime numbers between *125* and *170*;
- the *4th* order line exceeds that of the prime numbers between *1809* and *1814*;
- the last order, or $\boldsymbol{\Omega}(n)$, (in this case the eleventh, because $\log_2 3500 = 11.77$) contains either *1* or *2* elements and precisely

$$floor(n/(3 \cdot 2^{\boldsymbol{\Omega}(n)-1}))+1$$

More evident, in Fig.5 ($n = 16384$) we can see how also $\boldsymbol{\Omega}_5(m)$ tends to exceed $\boldsymbol{\Omega}_1(m)$ while $\boldsymbol{\Omega}_3(m)$ exceeds $\boldsymbol{\Omega}_2(m)$.

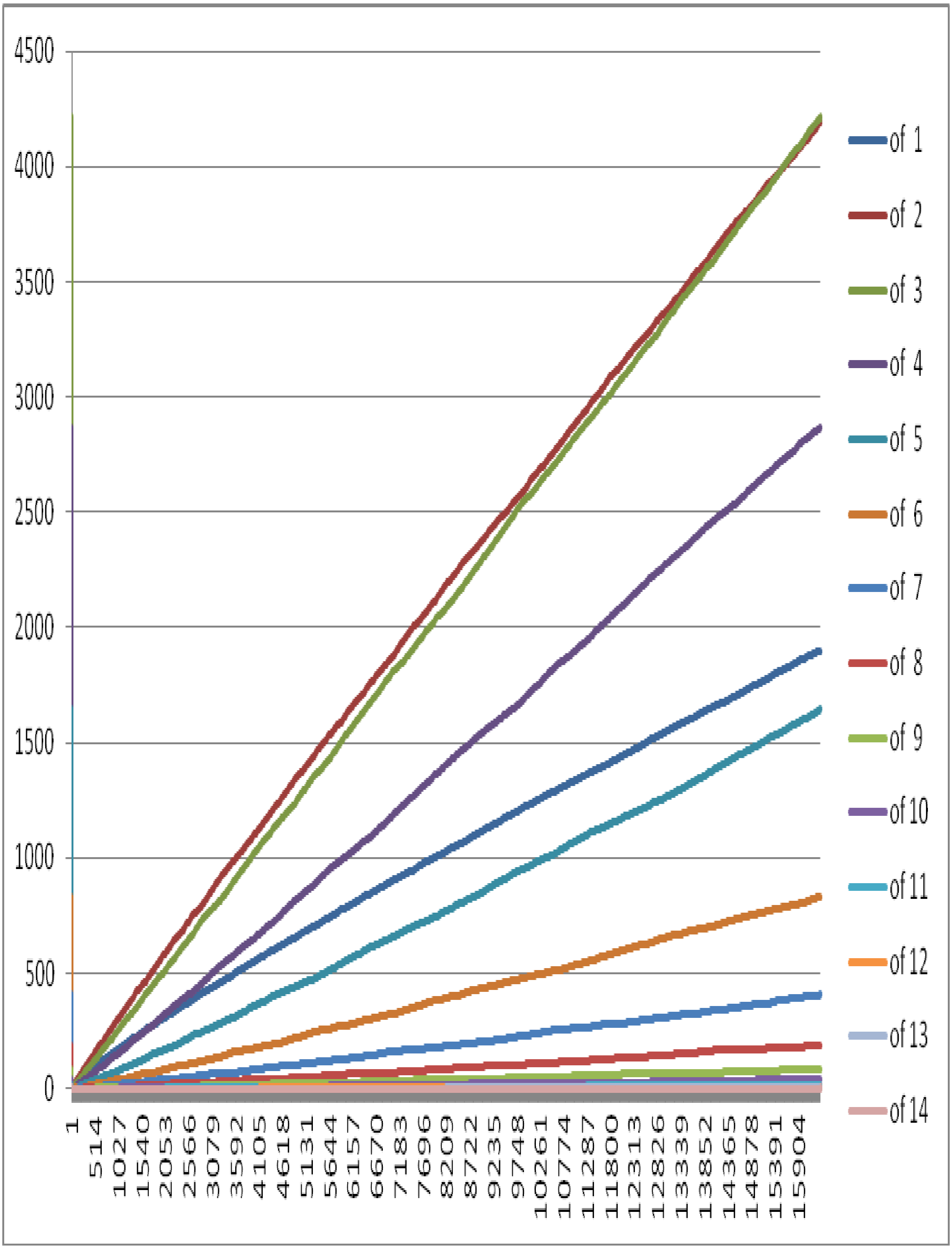
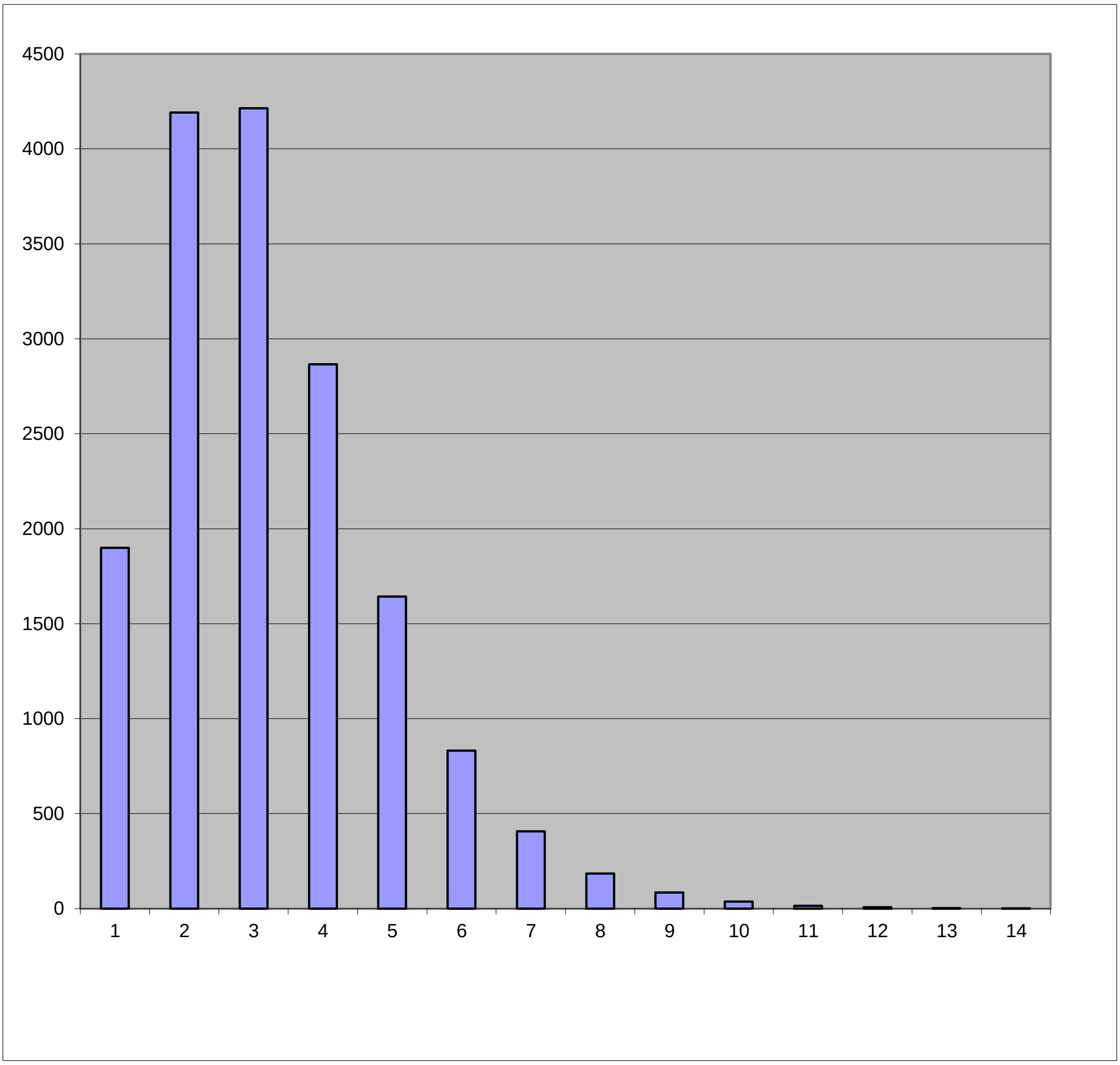


Fig. 5

The graph below shows the quantities relating to each of the *14* classes.



The cardinalities related to the classes *[k]* associated with each *o.f. k*, show a distribution that seems to resemble a distribution of Bernoulli of parameters to be determined.

x-axis: $\Omega_k(m)$ **y-axis: $\prod_k$**

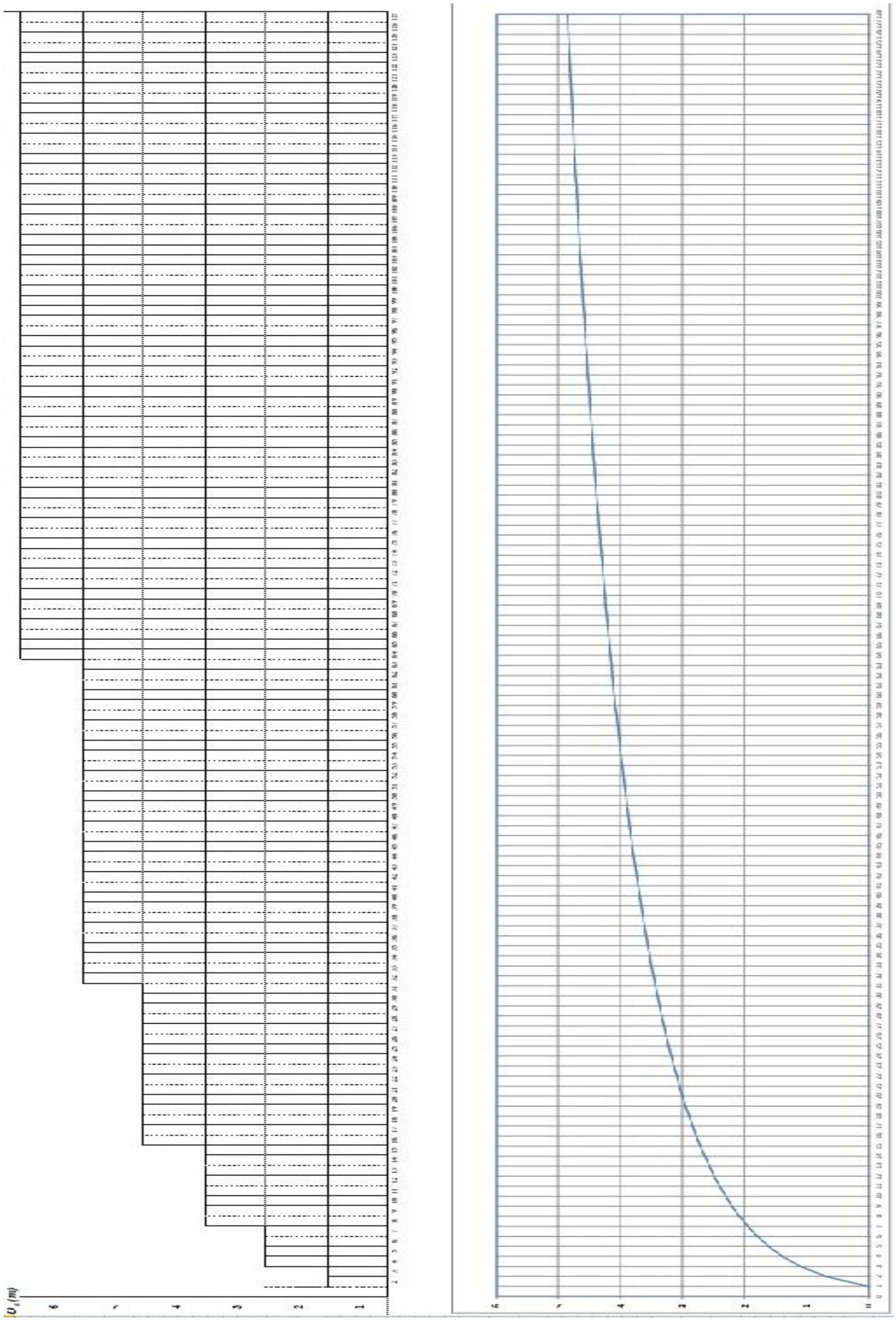


Fig. 6

3.1 Form and membership

From what has been said in **2. Representation**, $\forall$ integer $m \leq n$ the maximum *o.f.* possible is, obviously, $\Omega(m)=log_2 m$; this trend describes, as in Fig. 6, a logarithmic area $\int_{1,n} log_2 m \cdot dm \rightarrow ((1/ln(2)) \cdot \int_{1,n} ln(m) \cdot dm)$; also, assuming a uniform distribution, it is reasonable to suppose that

1) $$1/\boldsymbol{\Omega}(m)=1/\ log_2 m$$

indicates the "probability" that a number *m* belongs to the *i-th o.f.* $(1 \leq i \leq \boldsymbol{\Omega}_{Max}(n))$.

3.2 Calculations

From 1) it follows that the quantity of numbers theoretically belonging to the same *o.f.* *i* is

2) $$\prod_i(n)=\sum_{j=2^i,\dots,n} 1/\ log_2 j$$

The form 2), for *n* sufficiently large, could be written

3) $$\prod_i(n)_{n\rightarrow\infty} \equiv \int_{2^i-1,n} 1/log_2\ m \cdot dm \qquad \rightarrow$$
$$\rightarrow ln(2) \cdot \int_{2^i-1,n} 1/ln(m) \cdot dm$$

As for the quantity of numbers belonging to the *1st o.f.* (prime numbers), we may venture, based on Fig. 1, a method based on the comparison between the density of the numbers of the *1st o.f.* and the density of all numbers, in relation to the "area" respectively occupied:

q.ty numbers of the 1st o.f. / occupied area of the 1st level ≡ total q.ty numbers / total area occupied

replacing:

$$\prod_1(n) / (n-1) = (n-1) / ((1/ln(2)) \cdot \int_{1,n} ln(m) \cdot dm) \qquad \rightarrow$$

$$\rightarrow \quad \prod_1(n) = (n-1)^2 / ((1/ln(2)) \cdot \int_{1,n} ln(m) \cdot dm) \qquad \rightarrow$$

4) $$\rightarrow \quad \prod_1(n) = (n-1)^2 / ((1/ln(2)) \cdot (n \cdot ln(n) - n + 1))$$

In Tab. 3 this formula is compared with some famous formulas; even if the difference is considerable, it tends to stabilize.

	n	**numbers of primes *π(n)* between *2* and *n***	**Forecast with Gauss formula: *n/ln(n)***	**Deviation**	**Forevast with Legendre formula: *n/(ln(n)-1,08366)***	**Deviation**	**Forecast with formula: *(n-1)²/((1/ln(2))*(n*ln(n)-n+1))***	**Deviation**
1	1,E+01	4	4	0%	8	100%	4	0,0000%
2	1,E+02	25	22	-12%	28	12%	19	-24,0000%
3	1,E+03	168	145	-14%	172	2%	117	-30,3571%
4	1,E+04	1229	1086	-12%	1231	0,16%	844	-31,3263%
5	1,E+05	9592	8686	-9%	9588	-0,04%	6593	-31,2656%
6	1,E+06	78498	72382	-8%	78543	0,06%	54086	-31,0989%
7	1,E+07	664579	620421	-7%	665140	0,08%	458488	-31,0108%
8	1,E+08	5761455	5428681	-6%	5768004	0,11%	3978875	-30,9398%
9	1,E+09	50847534	48254942	-5%	50917519	0,14%	35143631	-30,8843%
10	1,E+10	455052511	434294482	-5%	455743004	0,15%	314697118	-30,8438%
11	1E+11	4118054813	3948131654	-4%	4124599869	0,16%	2849123470	-30,8139%
12	1E+12	37607912018	36191206825	-4%	37668527415	0,16%	26027810858	-30,7917%
13	1E+13	346065536839	334072678387	-3%	346621096885	0,16%	239564738510	-30,7747%
14	1E+14	3204941750802	3102103442166	-3%	3210012022164	0,16%	2219051528595	-30,7616%
15	1E+15	29844570422669	28952965460217	-3%	29890794226982	0,15%	20667038427351	-30,7511%
16	1E+16	279238341033925	271434051189532	-3%	279660033612131	0,15%	193393094398453	-30,7426%
17	1E+17	2623557157654230	2554673422960300	-3%	2627410589445920	0,15%	1817187896616470	-30,7357%
18	1E+18	24739954287740800	24127471216847300	-2%	24775244142175600	0,14%	17137370050627900	-30,7300%
19	1E+19	234057667276344000	228576043106975000	-2%	234381646366461000	0,14%	162143041302985000	-30,7252%
20	1E+20	2220819602560910000	2171472409516260000	-2%	2223801523570830000	0,13%	1538559370555550000	-30,7211%
22	1E+22	201467286689315906290	197406582683296000000	-2%	201721849105667000000	0,13%	139587362635261000000	-30,7146%

Best values

Tab. 3

Formula 4) is characterized by the constant *1/ln(2)*; if we put the variable *x* as the argument of the natural logarithm and we calculate on the basis of the effective values of $\prod(n)$ we see that $2 \rightarrow e$ and the value $1/ln(2) \rightarrow 1$

	n	**numbers of primes *π(n)* between *2* and *n***	**Determination of *x* by formula: *(n-1)²/((1/ln(x))*(n*ln(n)-n+1))***
1	1,E+01	4	1,998975129
2	1,E+02	25	2,514674058
3	1,E+03	168	2,703802757
4	1,E+04	1229	2,743583643
5	1,E+05	9592	2,741268962
6	1,E+06	78498	2,734624243
7	1,E+07	664579	2,731134516
8	1,E+08	5761455	2,728316364
9	1,E+09	50847534	2,726120217
10	1,E+10	455052511	2,724518942
11	1E+11	4118054813	2,723337991
12	1E+12	37607912018	2,722463310
13	1E+13	346065536839	2,721797198
14	1E+14	3204941750802	2,721278309
15	1E+15	29844570422669	2,720866441
16	1E+16	279238341033925	2,720534130
17	1E+17	2623557157654230	2,720262113
18	1E+18	24739954287740800	2,720036623
19	1E+19	234057667276344000	2,719847615
20	1E+20	2220819602560910000	2,719687615
22	1E+22	201467286689315906290	2,719433340

***e* =** 2,718282

Accordingly with the formula

4a) $\prod_1(n) = (n\text{-}1)^2 / (n\cdot ln(n) - n + 1))$ a better deviation is obtained:

	n	number of primes π(n) between 2 and n	Forecast with Gauss formula: n / ln (n)	Deviation	Forecast with Legendre formula: n / (ln (n) -1.08366)	Deviation	Forecast with formula: $(n\text{-}1)^2/(n*ln(n)\text{-}n+1)$	Deviation
1	1,E+01	4	4	0%	8	100%	6	50%
2	1,E+02	25	22	-12%	28	12%	27	8%
3	1,E+03	168	145	-14%	172	2%	169	0,59523809524%
4	1,E+04	1229	1086	-12%	1231	0,16%	1218	-0,89503661513%
5	1,E+05	9592	8686	-9%	9588	-0,04%	9512	-0,83402835696%
6	1,E+06	78498	72382	-8%	78543	0,06%	78030	-0,59619353359%
7	1,E+07	664579	620421	-7%	665140	0,08%	661459	-0,46947014576%
8	1,E+08	5761455	5428681	-6%	5768004	0,11%	5740304	-0,36711212706%
9	1,E+09	50847534	48254942	-5%	50917519	0,14%	50701542	-0,28711716875%
10	1,E+10	455052511	434294482	-5%	455743004	0,15%	454011971	-0,22866372009%
11	1E+11	4118054813	3948131654	-4%	4124599869	0,16%	4110416301	-0,18548835183%
12	1E+12	37607912018	36191206825	-4%	37668527415	0,16%	37550193650	-0,15347400295%
13	1E+13	346065536839	334072678387	-3%	346621096885	0,16%	345618860221	-0,12907284039%
14	1E+14	3204941750802	3102103442166	-3%	3210012022164	0,16%	3201414635781	-0,11005239082%
15	1E+15	29844570422669	28952965460217	-3%	29890794226982	0,15%	29816233849001	-0,09494716549%
16	1E+16	279238341033925	271434051189532	-3%	279660033612131	0,15%	279007258230820	-0,08275468270%
17	1E+17	2623557157654230	2554673422960300	-3%	2627410589445920	0,15%	2621647966812030	-0,07277107863%
18	1E+18	24739954287740800	24127471216847300	-2%	24775244142175600	0,14%	24723998785920000	-0,06449285086%
19	1E+19	234057667276344000	228576043106975000	-2%	234381646366461000	0,14%	233922961602470000	-0,05755234402%
20	1E+20	2220819602560910000	2171472409516260000	-2%	2223801523570830000	0,13%	2219671974013730000	-0,05167590136%
22	1E+22	201467286689315906290	197406582683296000000	-2%	201721849105667000000	0,13%	201381995844660000000	-0,04233483562%

Best values

Tab. 4a

	n	number of primes π(n) between 2 and n	Forecast with formula: $(n\text{-}1)^2/(n*ln(n)\text{-}n+1)$	Deviation	Forecast with formula: n/(ln(n)-1)	Deviation	Forecast with formula: Li()	Deviation
1	1,E+01	4	6	50%	8	100%	6	50%
2	1,E+02	25	27	8%	28	12%	30	20%
3	1,E+03	168	169	0,60%	169	0,60%	178	5,95238095238%
4	1,E+04	1229	1218	-0,90%	1218	-0,90%	1246	1,38323840521%
5	1,E+05	9592	9512	-0,83%	9512	-0,83%	9630	0,39616346956%
6	1,E+06	78498	78030	-0,60%	78030	-0,60%	78628	0,16560931489%
7	1,E+07	664579	661459	-0,47%	661459	-0,47%	664918	0,05100973699%
8	1,E+08	5761455	5740304	-0,37%	5740304	-0,37%	5762209	0,01308697195%
9	1,E+09	50847534	50701542	-0,29%	50701542	-0,29%	50849235	0,00334529498%
10	1,E+10	455052511	454011971	-0,23%	454011971	-0,23%	455055615	0,00068211908%
11	1E+11	4118054813	4110416301	-0,19%	4110416301	-0,19%	4118066401	0,00028139499%
12	1E+12	37607912018	37550193650	-0,15%	37550193650	-0,15%	37607950281	0,00010174189%
13	1E+13	346065536839	345618860221	-0,13%	345618860221	-0,13%	346065645810	0,00003148854%
14	1E+14	3204941750802	3201414635781	-0,11%	3201414635781	-0,11%	3204942065692	0,00000982514%
15	1E+15	29844570422669	29816233849001	-0,09%	29816233849001	-0,09%	29844571475288	0,00000352700%
16	1E+16	279238341033925	279007258230820	-0,08%	279007258230820	-0,08%	279238344248557	0,00000115121%
17	1E+17	2623557157654230	2621647966812030	-0,07%	2621647966812030	-0,07%	2623557165610820	0,00000030327%
18	1E+18	24739954287740800	24723998785920000	-0,06%	24723998785920000	-0,06%	24739954507236400	0,00000088721%
19	1E+19	234057667276344000	233922961602470000	-0,06%	233922961602470000	-0,06%	234057667376222000	0,00000004267%
20	1E+20	2220819602560910000	2219671974013730000	-0,05%	2219671974013730000	-0,05%	2220819602583650000	0,00000000102%
22	1E+22	201467286689315906290	201381995844660000000	-0,04%	201381995844660000000	-0,04%	201467286691247000000	0,00000000096%

Best values

Tab. 4b

Tab. 4b highlights how the formula *Li()* is certainly more precise than the formula 4a) used in Tab. 4a, which, while providing results similar to the simpler formula on its right, highlights an interesting particularity below. The results with the Riemann algorithm are left out: extremely better!

The proposed representation allows to see, in an evident way, the reason for the formulation of some important laws without the need for demonstrations.

As a result, the previous formulas would become:

1a) $$1/\Omega(m)=1/ln(m)$$

2a) $$\prod_i(n)=\sum_{j=2,\dots,n}^{i} 1/ln(j)$$

3a) $$\prod_i(n)_{n\to\infty}=\int_{2-1,n}^{i} 1/ln(m)\cdot dm \equiv Li(n)$$

(we obtain the formula of the integral Gauss logarithm)

4a) $$\prod_1(n) = (n-1)^2/(n\cdot ln(n)-n+1)$$

(dimensionally consistent with the laws of Legendre and Gauss. See Tab. 4a)

3.3 Evidence

The graph in Fig. 7a shows the progress of the function:

$y = n/x$ with $x \equiv (1, n)$ for $n = 29$

which represents an equilateral hyperbole; if we examine formula 4a) of the hypothesized "density" mode for determining the number of primes $\prod(n)$:

$$\prod(n) = (n - 1)^2/(n \cdot ln(n) - n + 1)$$

it is clear that the numerator $(n-1)^2$ is the area of the square with coordinates *(1,1) (1, n) (n, 1) (n, n)* and that the denominator $(n \cdot ln(n) - n + 1)$ is the area (*) delimited by the segment of equilateral hyperbole inscribed in the square and by the line $y = 1$; as a result we get one

geometric shape of $\prod(n)$

> *"The quantity Π(n) of prime numbers from 2 to n, with increasing n,* ***tends*** *to the ratio between the area of the square with side n-1 and the area delimited by the segment of equilateral hyperbole inscribed in the square and with the line y = 1"*

(*) $\int_{1,n} n/x \cdot dx = [n \cdot ln(x)]_{1,n} = n \cdot ln(n) \rightarrow$ area elimination $(1, n) \rightarrow n \cdot ln(n) - n + 1$

In this specific case $\prod(29) = 10$, while the formula provides *11* with a *10%* difference.

It should be noted in Fig. 7b that the area of the segment of equilateral hyperbole is equal to the area delimited by the logarithmic curve
$y = ln(x)$ with $x \equiv (1, n)$ for $n = 29$
and so:
$\int_{1,n} ln(x) \cdot dx = [x \cdot ln(x) - x]_{1,n} = n \cdot ln(n) - n + 1$

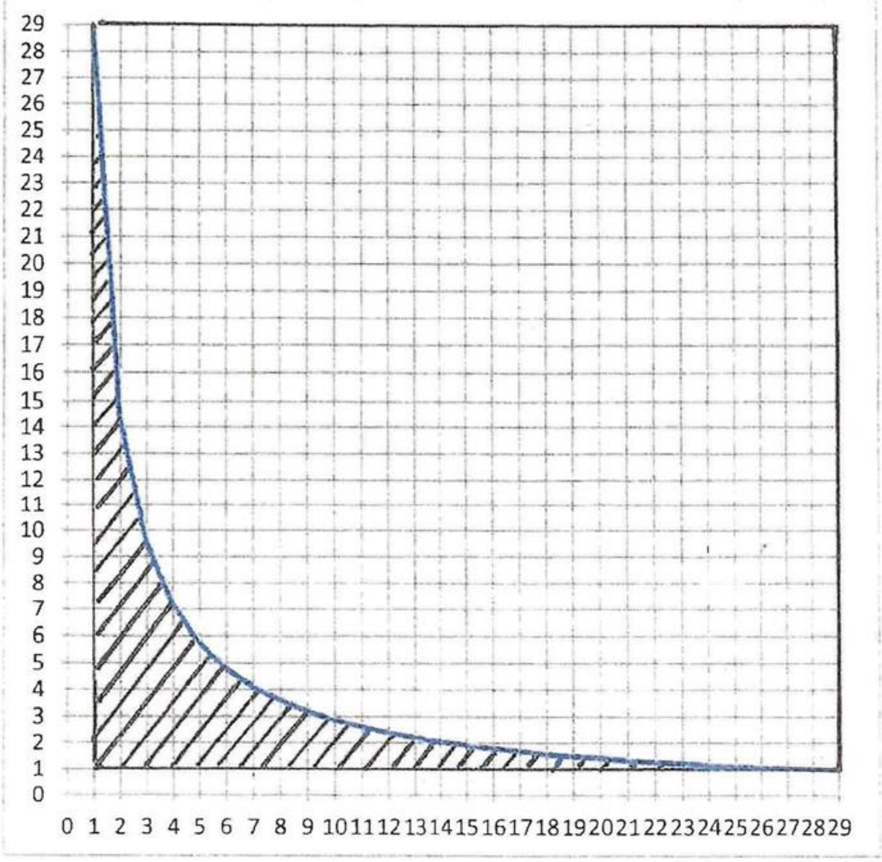

29 28 27 26 25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Fig. 7a

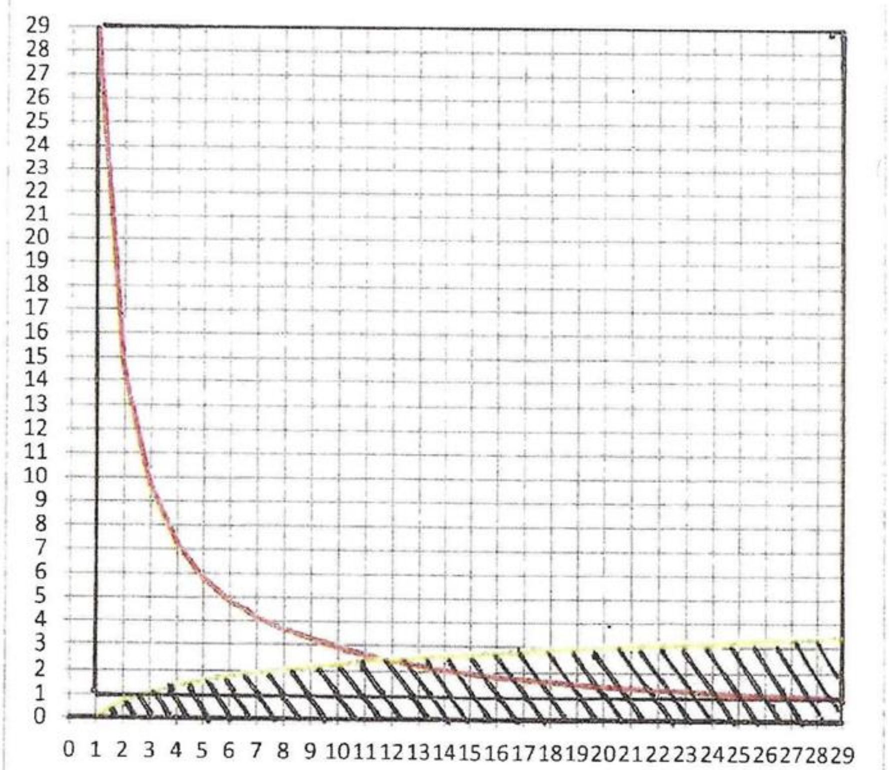

29 28 27 26 25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29

Fig. 7b

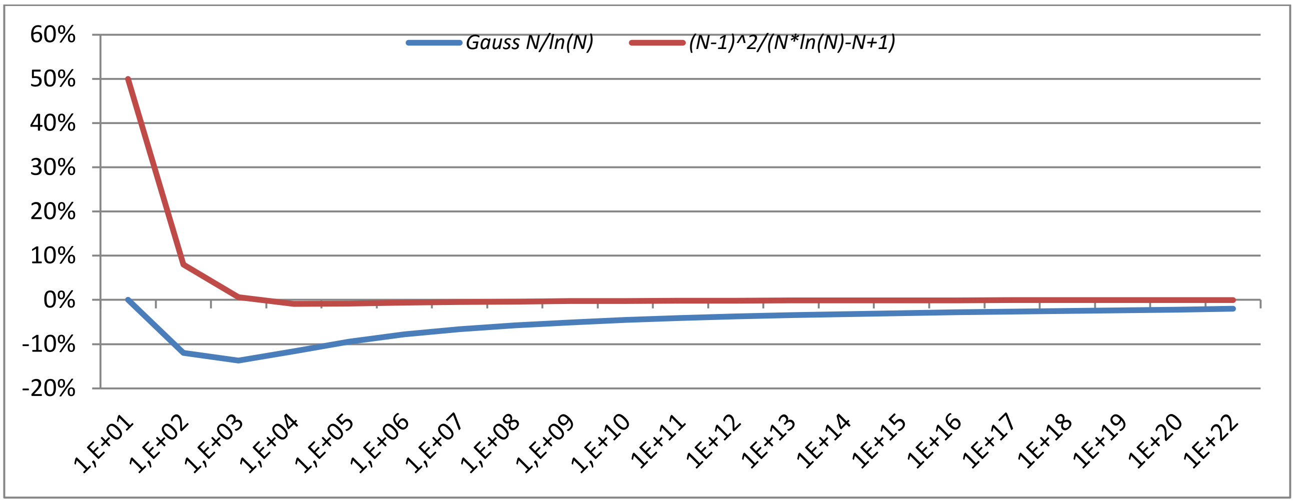


Fig. 8

	n	numbers of primes *π(n)* between *2* and *n*	Forecast with Legendre formula: *n/(ln(n)-1,08366)*	Deviation	Real trend of the Legendre constant
1	1,E+01	4	8	100%	-0,19741491
2	1,E+02	25	28	12%	0,60517019
3	1,E+03	168	172	2%	0,95537433
4	1,E+04	1229	1231	0,16%	1,07364387
5	1,E+05	9592	9588	-0,04%	1,08757100
6	1,E+06	78498	78543	0,06%	1,07633249
7	1,E+07	664579	665140	0,08%	1,07097559
8	1,E+08	5761455	5768004	0,11%	1,06395401
9	1,E+09	50847534	50917519	0,14%	1,05662871
10	1,E+10	455052511	455743004	0,15%	1,05036512
11	1E+11	4118054813	4124599869	0,16%	1,04512642
12	1E+12	37607912018	37668527415	0,16%	1,04087169
13	1E+13	346065536839	346621096885	0,16%	1,03734543
14	1E+14	3204941750802	3210012022164	0,16%	1,03437618
15	1E+15	29844570422669	29890794226982	0,15%	1,03184412
16	1E+16	279238341033925	279660033612131	0,15%	1,02966040
17	1E+17	2623557157654230	2627410589445920	0,15%	1,02775776
18	1E+18	24739954287740800	24775244142175600	0,14%	1,02608512
19	1E+19	234057667276344000	234381646366461000	0,14%	1,02460312
20	1E+20	2220819602560910000	2223801523570830000	0,13%	1,02328087
22	1E+22	201467286689316000000	201721849105667000000	0,13%	1,02102216

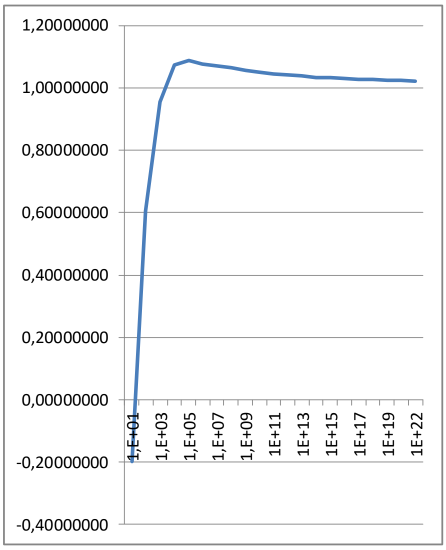


Fig. 9

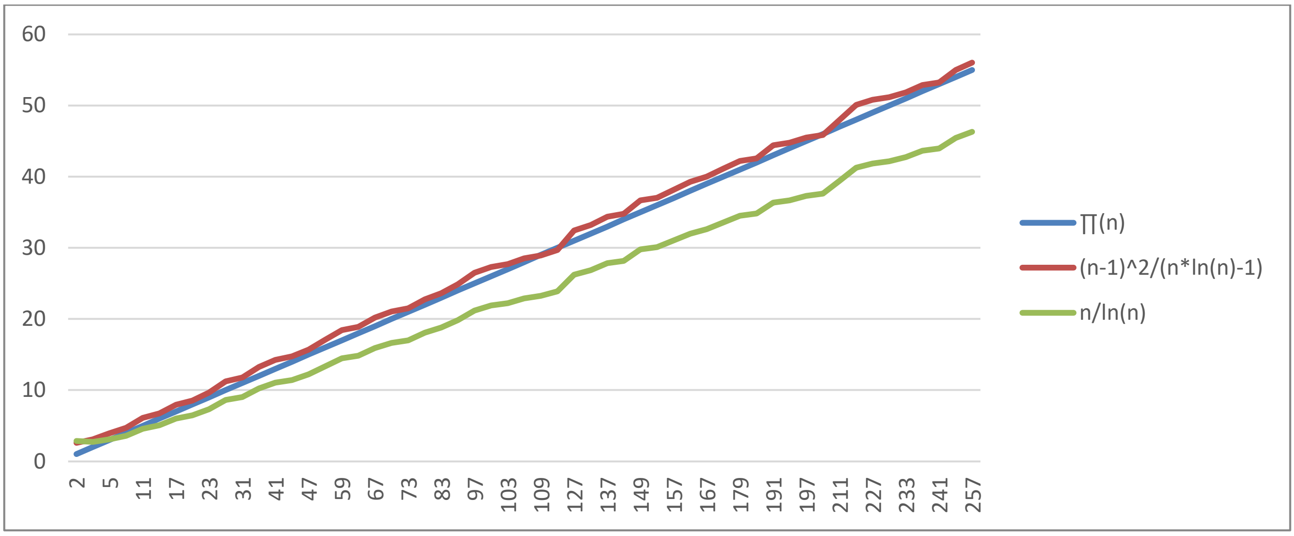


Fig. 10

In formula 4a) it is noted that this is dimensionally consistent with the formulas of Legendre and Gauss.

If we develop and divide numerator and denominator by n:

$$(n^2 - 2\cdot n + 1)/(n\cdot ln(n) - n + 1) \quad \rightarrow \quad (n - 2 + 1/n)/(ln(n) - 1 + 1/n)$$

and apply the limit for $n\rightarrow\infty$:

$$lim_{n\rightarrow\infty} \; (n - 2 + 1/n)/(ln(n) - 1 + 1/n) \quad \rightarrow \quad n/ln(n)$$

Fig. 8 compares the trend of the gap between the two formulations.

Fig. 9 shows the trend of the Legendre's constant.
Legendre found better results than those of the Gauss formula applying the value *-1.08366*, however, not being able to have today's technologies to perform calculations, he had to limit his reach. The graph could suggest a value tending to *1* for the Legendre's constant.

Fig. 10 compares the real quantity of prime numbers with the formula 4a) and with the Gauss formula.

